Exact Solutions and Localized Excitations of Burgers System in (3+1) Dimensions

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With the help of a Cole-Hopf transformation, the nonlinear Burgers system in (3+1) dimensions is reduced to a linear system. Then by means of the linear superposition theorem, a general variable separation solution to the Burgers system is obtained. Finally, based on the derived solution, a new type of localized structure, i.e., a solitonic bubble is revealed and some evolutional properties of the novel localized structure are briefly discussed.

Key words: Cole-Hopf Transformation; (3+1)-Dimensional Burgers System; Solitonic Bubble; Evolutional Behaviour.

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1. Introduction

In nonlinear science, soliton theory plays an important role and has been widely applied in many natural sciences such as biology, chemistry, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, optics, plasma physics, field theory, condensed matter physics, and so on [1 – 12]. As is known, many dynamical systems in natural fields are characterized by nonlinear evolution partial differential equations called as governing equations. Solving these nonlinear equations is much more difficult than solving the linear ones. Fortunately, a wealth of approaches for finding exact solutions of nonlinear partial differential equations, such as the inverse scattering transformation [13], bilinear approach [14], symmetry method [15], Bäcklund and Darboux transformations [16], Painlevé truncated expansion [17], variable separation approach [18], and extended mapping approach [19, 20], etc. were presented successfully, and applied to many integrable systems in (1+1) dimensions and (2+1) dimensions. Some significant types of localized excitations such as dromions, rings, compactons, peakons, and folded solitons are derived [19 – 25]. Now an important and natural problem is that: can we obtain some localized excitations, particularly some novel localized excitations like solitonic bubbles [26 – 31], taper-like solitons in higher-dimensional systems? To answer the question, we take the (3+1)-dimensional Burgers system as a concrete example:

$$u_t = 2uu_y + 2vu_x + 2wu_z + u_{xx} + u_{yy} + u_{zz},$$
$$u_x = v_y, \quad u_z = w_y, \tag{1}$$

which is a generalized version in the (3+1) dimensions of the Burgers system. Obviously, if u is z-independent (or $z = x$, $w = u$), (1) will be degenerated to the known (2+1)-dimensional Burgers system, which is derived from the generalized Painlevé integrability classification in [32] and has been proved to be variable separation approach solvable [33 – 35]. Furthermore, if u is both z-independent and y-independent (or $z = y = x$, $w = v = u$), (1) will become a usual (1+1)-dimensional Burgers equation, which has been widely applied in many scientific fields. An alternative potential version of (1) was obtained from the invertible deformation of a heat conduction equation [36]. More detail physical backgrounds about the (3+1) dimensions of the Burgers system can be refereed to the above mentioned literature and reference therein. In the following parts of the paper, we discuss its general exact solutions with interesting localized excitations such as solitonic bubble, taper-like solitons and their evolution properties via a Cole-Hopf transformation approach.

2. Exact Solutions to the (3+1)-Dimensional Burgers System

In this section, we give out a quite general solution to the (3+1)-dimensional Burgers system. As is known, to search for solitary wave solutions to a given nonlinear partial differential system, one may use different approaches, such as multilinear variable separation approach [18], Painlevé truncated expansion method [37], and the mapping approach [38]. One of the useful and powerful methods is the so-called Cole-Hopf transformation approach [39]. Let us begin with a special Cole-Hopf transformation for u, v, w in (1):

$$u = (\ln f)_y + u_0, \quad v = (\ln f)_x + v_0, \quad w = (\ln f)_z + w_0, \tag{2}$$

where $f = f(x,y,z,t)$ is an arbitrary function of variables $\{x,y,z,t\}$ to be determined and $\{u_0, v_0, w_0\}$ is an arbitrary known seed solution of (1). It is evident that (1) possess a trivial seed solution

$$u_0 = 0, \quad v_0 = v_0(x,z,t), \quad w_0 = 0, \tag{3}$$

with $v_0(x,z,t)$ being an arbitrary function of the indicated arguments. Now substituting (2) together with the seed solution (3) into (1) yields

$$(f_y - f\partial_y)(f_t - f_{xx} - f_{yy} - f_{zz} - 2v_0 f_x) = 0. \tag{4}$$

Based on (4), one can find that if f satisfies

$$f_t - f_{xx} - f_{yy} - f_{zz} - 2v_0 f_x = 0, \tag{5}$$

then (4) is satisfied automatically.

In [40], Zhu and Zheng take f as such an ansatz $f = \chi(x,z,t) + \varphi(y,t)$ and derive some special solutions for the Burgers system via a mapping approach. In [34], motivated by some works on (2+1)-dimensional cases, Ying and Lou suppose that f has the following variable separation solution $f = a_0 + a_1 p(x,z,t) + a_2 q(y,t) + a_3 p(x,z,t)q(y,t)$, which is essentially equivalent to a modified Hirota's multisoliton form when p and q are chosen appropriate exponential functions [18]. In our present paper, we try to obtain a more general solution to the Burgers system by choosing a more general form for f. Since (5) is a linear equation, one can naturally take advantage of the linear superposition theorem. For instance

$$f = \lambda + \sum_{k=1}^{N} P_k(x,z,t)Q_k(y,t), \tag{6}$$

where λ is an arbitrary constant, $P_k(x,z,t) \equiv P_k$ and $Q_k(y,t) \equiv Q_k$ $(k = 1,2,\cdots,N)$ are variable separated functions of $\{x,z,t\}$ and $\{y,t\}$, respectively. It is obvious that the prevenient mentioned ansatz [34, 35, 40] is a special case of the general ansatz (6).

Inserting the ansatz (6) into (5) yields following set of variable separation equations:

$$P_{kt} - 2v_0 P_{kx} - P_{kxx} - P_{kzz} + \Gamma_k(t)P_k = 0, \tag{7}$$

$$Q_{kt} - Q_{kyy} - \Gamma_k(t)Q_k = 0, \tag{8}$$

where $\Gamma_k(t), (k = 1,2,\cdots,N)$, are arbitrary functions of time t. Then a general variable separation excitation for the Burgers system (1) reads

$$u = \frac{\sum_{k=1}^{N} P_k Q_{ky}}{\lambda + \sum_{k=1}^{N} P_k Q_k}, \tag{9}$$

$$v = \frac{\sum_{k=1}^{N} P_{kx} Q_k}{\lambda + \sum_{k=1}^{N} P_k Q_k} + v_0, \tag{10}$$

$$w = \frac{\sum_{k=1}^{N} P_{kz} Q_k}{\lambda + \sum_{k=1}^{N} P_k Q_k}, \tag{11}$$

where v_0, P_k, and Q_k admit (7) and (8).

Considering the complexity of (9), (10), and (11) and for the convenience of the following discussions, we make simplifications further and give out some special exact solutions.

Case 1. We first consider a simplest case: $N = 1$, $\{P_1,\ Q_1\} = \{P(x,z,t), Q(y,t)\}$, $\Gamma_1(t) = \tau(t)$. Then (6), (7), and (8) become

$$f = \lambda + PQ, \tag{12}$$

$$P_t - P_{xx} - 2v_0 P_x - P_{zz} + \tau(t)P = 0, \tag{13}$$

$$Q_t - Q_{yy} - \tau(t)Q = 0. \tag{14}$$

It is easy to obtain a general solution of (13). Since $v_0(x,z,t)$ is an arbitrary seed solution, we can view P as an arbitrary function of $\{x,z,t\}$, then the seed solution v_0 is fixed by (13),

$$v_0 = \frac{P_t - P_{xx} + P_{zz} + \tau(t)P}{2P_x}. \tag{15}$$

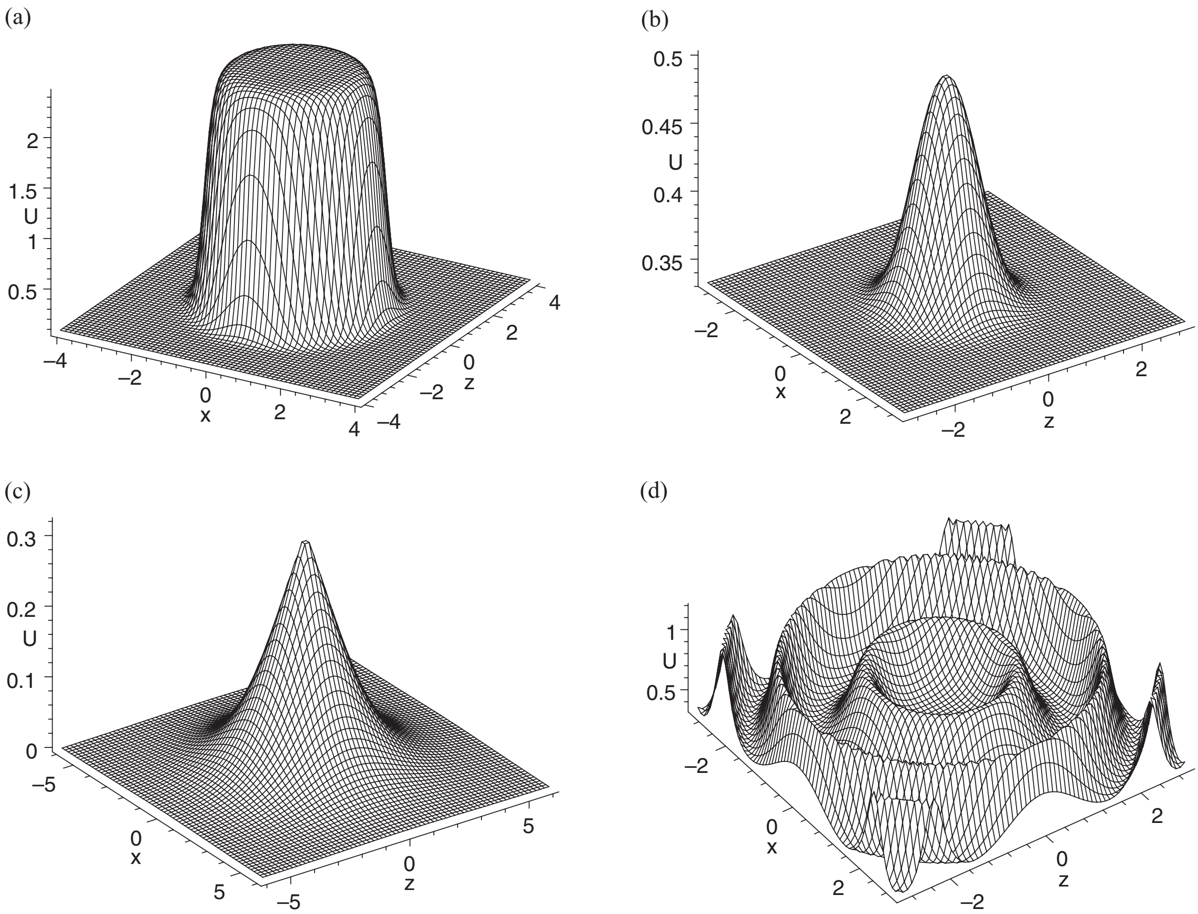

Fig. 1. (a) Profile of plateau-type ring soliton for the field U given by (28) with the condition (29); (b) Plot of standard dromion for the field U given by (28) with the condition (30); (c) Profile of taper-like soliton for the field U given by (28) with the condition (31); (d) Plot of the periodic wave excitation for the field U given by (28) with the condition (32), and the parameters fixed $a_1 = a_2 = k = 1$, $a_0 = a_3 = 0$.

As to the linear heat equation (14), its general solution has the following form:

$$Q(y,t) = \sum_{i=0}^{\infty}\left[C_i \mathrm{e}^{A_i(y+A_i t)} + D_i \mathrm{e}^{A_i(y-A_i t)}\right] \cdot \exp \int^t \tau(s)\mathrm{d}s, \tag{16}$$

where A_i, C_i, and D_i are arbitrary constants.

Finally, we derive a special variable separation solution for the Burgers system (1):

$$u_1 = \frac{PQ_y}{\lambda + PQ}, \tag{17}$$

$$v_1 = \frac{P_x Q}{\lambda + PQ} + \frac{P_t - P_{xx} + P_{zz} + \tau(t)P}{2P_x}, \tag{18}$$

$$w_1 = \frac{P_z Q}{\lambda + PQ}, \tag{19}$$

with an arbitrary function $P(x,z,t)$ and $Q(y,t)$ shown by (16).

Case 2. In a similar way, we consider another case: $N = 3$, $\lambda = a_0$, $\{P_1,\ Q_1\} = \{p(x,z,t), a_1\}$, $\{P_2,\ Q_2\} = \{a_2,\ q(y,t)\}$, $\{P_3,\ Q_3\} = \{p(x,z,t),\ a_3 q(y,t)\}$, $\Gamma_k(t) = 0$, $(k = 1,2,3)$, here a_i, $(i = 0\ldots 3)$, are arbitrary constants, then (6), (7), and (8) become

$$f = a_0 + a_1 p + a_2 q + a_3 pq, \tag{20}$$

$$p_t - p_{xx} - 2v_0 p_x - p_{zz} = 0, \tag{21}$$

$$q_t - q_{yy} = 0. \tag{22}$$

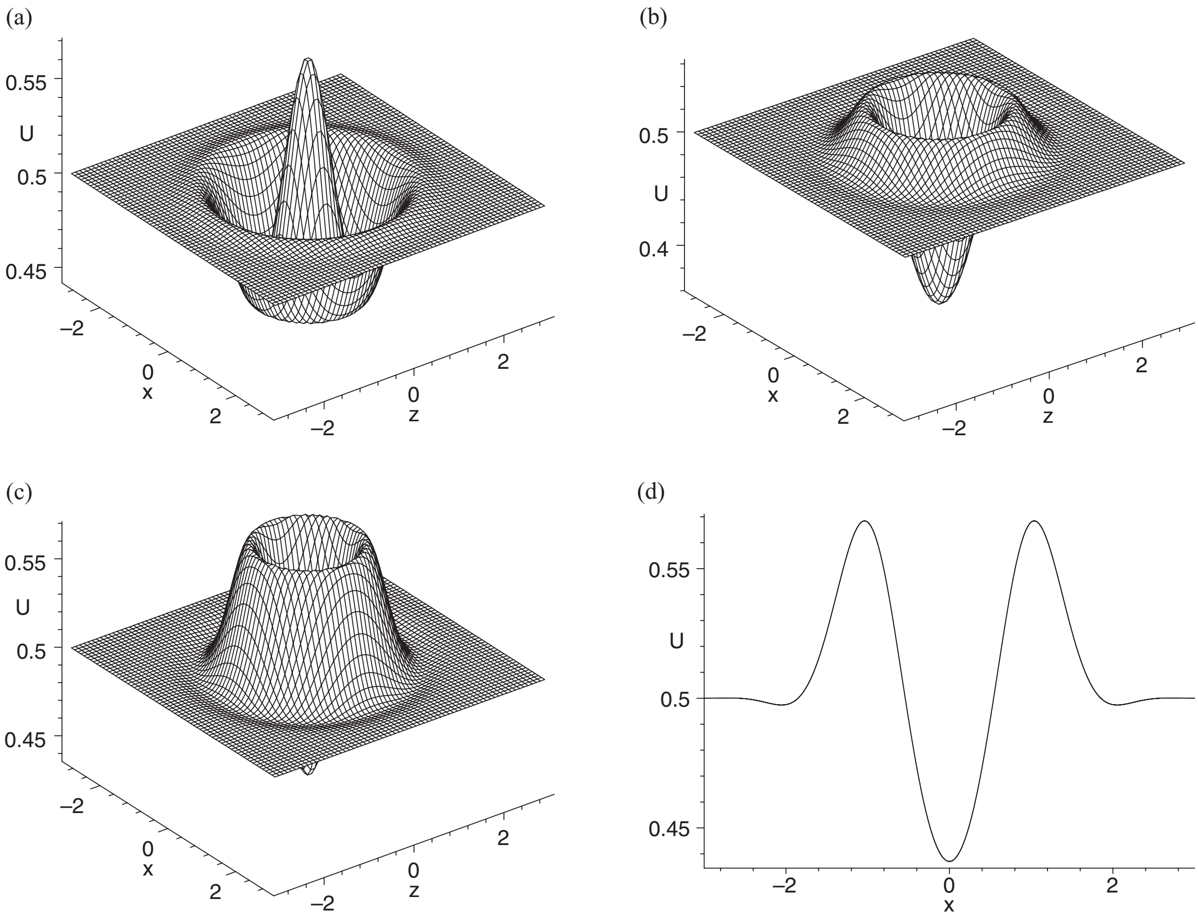

Fig. 2. Evolutional plots of a solitonic bubble for the field U given by (28) with the condition (33) at times: (a) $t = 0.5$, (b) $t = 2$, (c) $t = 4$. (d) Sectional view related to (c) at $z = 0$.

Based on (20), (21), and (22), one can obtain another special exact solution for the Burgers system (1):

$$u_2 = \frac{(a_2 + a_3 p)q_y}{a_0 + a_1 p + a_2 q + a_3 pq}, \tag{23}$$

$$v_2 = \frac{(a_1 + a_3 q)p_x}{a_0 + a_1 p + a_2 q + a_3 pq} + \frac{p_t - p_{xx} - p_{zz}}{2p_x}, \tag{24}$$

$$w_2 = \frac{(a_1 + a_3 q)p_z}{a_0 + a_1 p + a_2 q + a_3 pq}, \tag{25}$$

with an arbitrary function $p(x,z,t)$ and $q(y,t)$ expressed by

$$q(y,t) = \sum_{i=0}^{\infty}\left[C_i e^{A_i(y+A_i t)} + D_i e^{A_i(y-A_i t)}\right] + \sum_{j=0}^{\infty}\left[K_j\left(y^2 + 2K_j t\right) + L_j y + B_j\right], \tag{26}$$

where A_i, B_j, C_i, D_i, K_j, and L_j are all arbitrary constants.

It is interesting to mention that the previous derived result in [40] is equivalent to a special solution of *Case 2* when setting $a_3 = a_0 = 0$, $a_2 = a_1 = 1$, and $p = \chi(x,z,t)$, $q = \varphi(y,t)$, i.e., $u = \frac{\varphi_y}{\chi+\varphi}$, $v = \frac{\chi_x}{\chi+\varphi} + \frac{\chi_t - \chi_{xx} - \chi_{zz}}{2\chi_x}$, and $w = \frac{\chi_z}{\chi+\varphi}$.

3. Some Novel Localized Excitations in the Burgers System

In this part, we reveal some interesting localized coherent structures for the Burgers system. The intrusion of the (3+1)-dimensional arbitrary function $p(x,z,t)$ and the function $q(y,t)$ (a solution of the linear heat

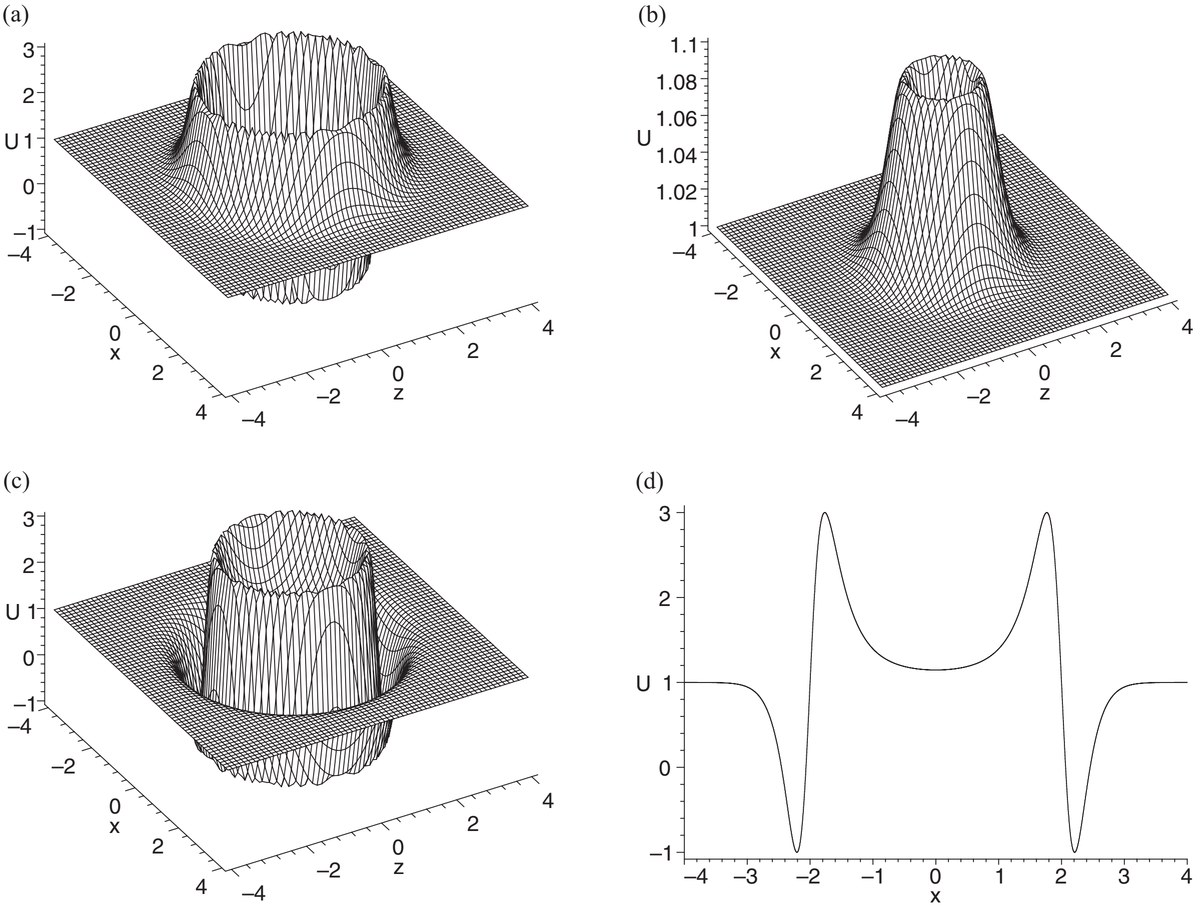

Fig. 3. Time evolutional profiles of a solitonic bubble for the field U given by (28) with the condition (34) and the parameters fixed $a_1 = a_2 = k = 1$, $a_0 = a_3 = 0$ at times: (a) $t = -2$, (b) $t = 0$, (c) $t = 2$. (d) Sectional view related to (c) at $z = 0$.

equation) in the above solutions implies that the physical fields u, v, and w or their potentials may possess rich localized structures. In [34, 35], the authors discussed some localized excitations of a potential $\mathcal{R} \equiv 2u_x = 2v_y$, i.e.,

$$\mathcal{R} = \frac{2(a_1a_2 - a_0a_3)q_yp_x}{(a_0 + a_1p + a_2q + a_3pq)^2}. \tag{27}$$

Comparing the special potential $\mathcal{R}$ expressed by (27) with the so called common formula (1.1) in [18], one can find they possess the completely same form. The only differences are that the function q is a solution of a linear heat equation and p is an arbitrary function of three independent arguments. Therefore, similar to the (2+1)-dimensional cases, some special localized excitations based on the common formula may be re-derived in the (3+1)-dimensional Burgers system. Since these localized structures have been reported in the previous literature [34, 35], we neglect the related discussions in this section.

However, as far as we know, the physical field u of the Burgers system is little discussed in previous literature. For convenience, here we do not study the general field u (9) but only discuss the special field u_2 expressed by (23), i.e.,

$$U \equiv u_2 = \frac{(a_2 + a_3p)q_y}{a_0 + a_1p + a_2q + a_3pq}. \tag{28}$$

From (28), we do know that for general selections of the functions p and q, there may be some singularities for the field U. We have to choose the functions p and q carefully to avoid these singularities. However,

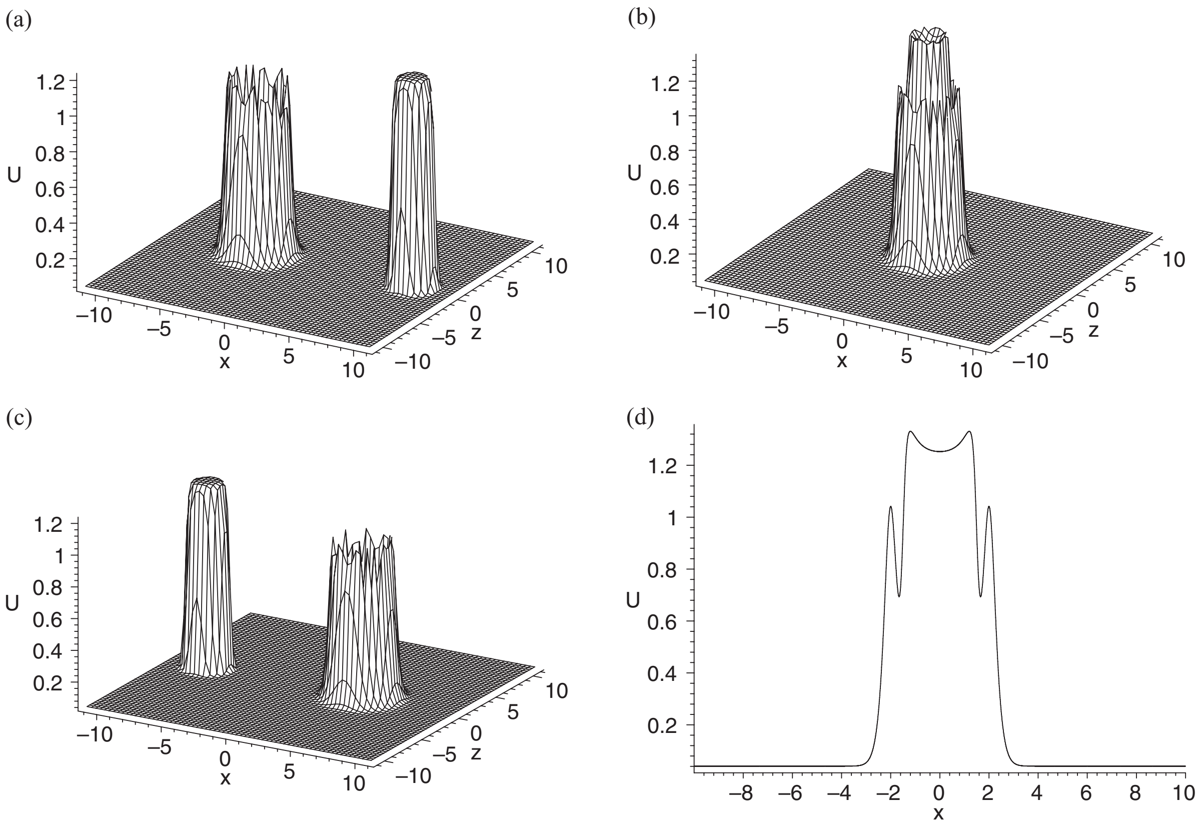

Fig. 4. Time evolutional profiles of the interaction between a solitonic bubble and a plateau-type ring soliton for the field U given by (28) with the condition (35) and the parameters fixed $a_1 = a_2 = k = 1$, $a_0 = a_3 = 0$ at times: (a) $t = -4$, (b) $t = 0$, (c) $t = 4$. (d) Sectional view related to (b) at $z = 0$.

even in the special situation, one can still find rich localized structures for the (3+1)-dimensional Burgers system by selecting the functions p and q appropriately. For example, when we consider $q = ky$ and set p to be

$$p^{-1} = 3 - \exp[\tanh(x^2 + z^2 + t^2)], \tag{29}$$

$$p = 1 + \tanh(x^2 + z^2 + t^2), \tag{30}$$

$$p = 1 + \exp\sqrt{(x^2 + z^2 + t^2)}, \tag{31}$$

$$p = 1 + \tanh(x^2 + z^2)\sin(x^2 + z^2 + t^2), \tag{32}$$

respectively, then we can obtain a plateau-type ring soliton, a standard dromion excitation, a taper-like soliton, and a periodic wave excitation for the physical field U (28) shown by Figure 1.

It should be mentioned that in Figure 1, which is similar to the following cases to be discussed, we are taking $y = 0$ section of the solution; as a result the solution looks localized on the (x,z) plane. One may ask: Are these solutions localized in three dimensions, i.e. when considered as functions of x, z, and y? The answer is positive as we choose the parameters of function $q(y,t)$ shown by (26) appropriately. The similar cases have be reported in [34, 35].

3.1. Solitonic Bubbles

In the following discussions, we will focus our attention on a novel type of localized structure which may exist in certain situations. In [26–31], the authors have reported some solitonic bubbles. Actually, these localized excitations also exist in the Burgers system. For instance, when choosing $q = ky$ and setting p to be

$$p = 1 + \exp(x^2 + z^2)\sin(x^2 + z^2 - t^2), \tag{33}$$

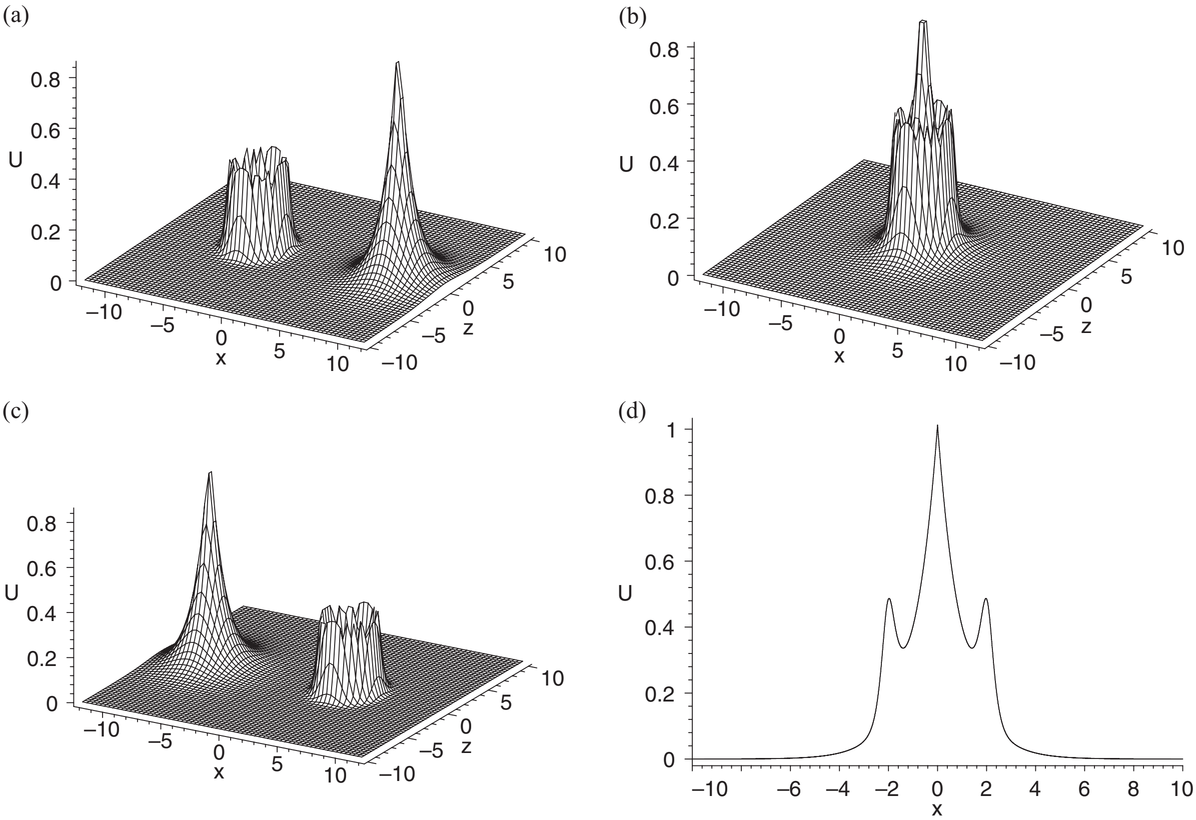

Fig. 5. Time evolutional profiles of the interaction between a solitonic bubble and a taper-like soliton for the field U given by (28) with the condition (36) and the parameters fixed $a_1 = a_2 = k = 1$, $a_0 = a_3 = 0$ at times: (a) $t = -4$, (b) $t = 0$, (c) $t = 4$. (d) Sectional view related to (b) at $z = 0$.

then we can derive a solitonic bubble for the physical field U expressed by (28) with the fixed parameters $a_1 = a_2 = k = 1$, $a_0 = a_3 = 0$. The time evolutional profiles of the solitonic bubble are shown by Figure 2. From this figure, one can find the solitonic bubble with periodic behaviour since the bubble vibrates periodically. At initial time, the amplitude of the bubble moves along the negative direction of the field U shown by Figure 2(a). With time elapsing, the amplitude of the bubble becomes smaller and smaller till to a minimal amplitude presented in Figure 2(b). To the time of Figure 2(c), the amplitude of the bubble evolves along the opposite direction, i.e., along the positive direction of the field U.

Similarly, if taking $q = ky$ and p to be

$$p^{-1} = 1 + (\sin(x^2 + z^2 - t^2))_t, \tag{34}$$

then we can derived another type of solitonic bubble also with periodic behaviours for the physical field U (28) presented in Figure 3.

3.2. Interactions among Solitonic Bubble and other Solitons

Now let us pay our attention to the interaction among solitonic bubble and other solitons. First, we consider a simple case: an interaction between a solitonic bubble and a plateau-type soliton. For example, when $q = ky$ and p is chosen to be

$$p^{-1} = 2 + \operatorname{sech}[(x-t)^2 + z^2 - 4] - 0.5\exp[\tanh((x+2t)^2 + z^2 - 4)], \tag{35}$$

then we derive a combined excitation with a solitonic bubble and a plateau-type soliton for the physical field

U (28) presented in Figure 4. From this figure and by detailed analysis, we find the interaction between the solitonic bubble and the plateau-type soliton is completely elastic since their amplitudes, velocities, and wave shapes do not undergo any change after their collision. The elastic collision is a trivial interaction, which follows simply from the fact that (35) provides the exact solution both for t equal to plus and minus infinity. In other words, the elastic collision solution is effectively a solution to a linear equation; due to the linear superposition principle there cannot be any interaction between terms in the solution. The same reason applies to the next case (Fig. 5).

Along with the above line, we consider another case: the interaction between a solitonic bubble and a taper-like soliton. If $q = ky$ and p is set to be

$$p^{-1} = 0.5\mathrm{sech}[(x-t)^2 + z^2 - 4] + \exp\left[-\sqrt{((x+2t)^2 + z^2)}\right], \tag{36}$$

then we can derive another type of combination excitation with a solitonic bubble and a taper-like soliton for the physical field U (28) shown by Figure 5. Similar to the case of Figure 4, one can find the interaction between the solitonic bubble and the taper-like soliton is also completely elastic since their amplitudes, velocities, and wave shapes are completely preserved after their collision. It should be mentioned that the completely elastic behaviours occurred in Figures 4 and 5 are rather determined by the selections of the function p: (35) and (36).

4. Summary and Discussion

In summary, starting from a special Cole-Hopf transformation, the nonlinear Burgers system in (3+1) dimensions is reduced to a linear system. Then by means of the linear superposition theorem, a general variable separation solution to the (3+1)-dimensional Burgers system is successfully obtained. Based on the derived variable separation solutions with arbitrary characteristic functions, we obtain some localized excitations such as solitonic bubbles and taper-like solitons for the Burgers system. Meanwhile, some evolutional properties of solitonic bubbles and interactions among a solitonic bubble and other type solitons are briefly discussed. We expect the solitonic bubbles may be useful in future studies for the intricate nature world. To the best of our knowledge, the solitonic bubbles, the taper-like solitons, and their evolutional properties for (3+1)-dimensional Burgers system were little reported in previous literature. However, our present short note is merely an initial work. Due to widely potential applications of soliton theory, more studied on the new localized excitations and their related evolutional properties, particularly their applications in reality should be performed further.

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